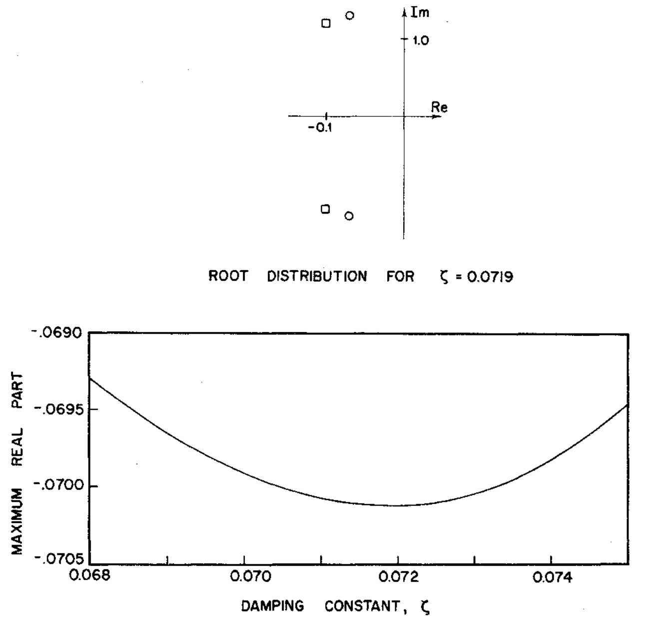

Fig. 2 Behavior of maximum real part in the neighborhood of optimal solution.

where δ is a positive constant. Details on these root distributions are discussed elsewhere.[3,4] Of these ten root configurations, the optimal condition used by Amieux and Dureigne,[1] namely $a_1{}^* = 0$, represents an a priori restriction to one of the four types (III, VII, VIII, X); the other possibilities are discarded at the outset. Their condition is therefore in error since unless there are further constraints on the A_i ($i = 1, \ldots, 4$), all types are possible.

On the other hand, suppose that the true optimal solution corresponds to one of the four types (III, VII, VIII, X). Then their analytical solutions can be derived (exactly) without difficulty. With $a, b, c, d, e, \rho, r, \omega^2$ as defined by Amieux and Dureigne[1] and with $A_1 = (1+\rho b)/(1+\rho a)$, $A_2 = (r+\omega^2+\rho c)/(1+\rho a)$, $A_3 = (\omega^2+\rho d)/(1+\rho a)$, $A_4 = (\omega^2 r+\rho e)/(1+\rho a)$, then the results can be established as described in Table 1.

Table 1 Conditions on A_i ($i = 1, \ldots, 4$) for optimal solution q^*

Type of optimal solution	Optimal solution $= q^*$	Conditions on $A_i, i = 1, \ldots, 4$
Amieux and Dureigne	$q^* = 2[\rho(-a\omega^4+c\omega^2-e)]^{1/2}/\omega$	Nil
Type III	$q^* = 2[r-\omega^2+\rho(ar+br+c-2d+b\omega^2-3a\omega^2)+\rho^2(ab(r+\omega^2)+c(a+b)-2a^2\omega^2-4ad)+\rho^3(abc-2a^2d)]/(1+\rho b)^{3/2}$	$2A_3 < A_1A_2 < 6A_3$; $16A_1{}^3A_4+A_1{}^2A_2{}^2-12A_1A_2A_3+20A_3{}^2 = 0$; $A_1{}^2A_2{}^2+2A_3{}^2-2A_1A_2A_3-2A_4A_1{}^2 > 0$
Type VII	Same as Type III	$2A_3 < A_1A_2 < 6A_3$; $16A_1{}^3A_4+A_1{}^2A_2{}^2-12A_1A_2A_3+20A_3{}^2 = 0$
Type VIII	$q^* = 2[1+\rho a]\cdot[(1+\rho b)^2\cdot(\omega^2(1-r)+\rho(d-e)-(1+\rho a)\cdot(\omega^4+\rho^2d^2+2\rho d\omega^2)]/[(1+\rho b)^{3/2}\cdot(\omega^2+\rho d)^{1/2}]$	$A_3-A_4-A_3{}^2/A_1{}^2 > 0$; $A_3{}^2-A_1A_2A_3+A_1{}^2(A_3-A_4) = 0$
Type X	$q^* = 4(1+\rho a)\cdot(\omega^2+\rho d)^{1/2}/(1+\rho b)^{3/2}$	$A_1A_2-6A_3 = 0$; $A_1{}^2A_3-A_4A_1{}^2-A_3{}^2 = 2A_1A_3$

It can be seen that none of the exact solutions reduces to Amieux and Dureigne's[1] result. The discrepancy indeed comes from the following three approximations: 1) the characteristic equation is approximated by a truncated Taylor series; 2) the real part of the approximated characteristic equation is again approximated; and 3) the imaginary part of the approximated characteristic equation is not satisfied by the solution to 2).

The discrepancy can be illustrated by a numerical example of their ball-in-tube system. For $m = 0.150$ Kg, $\omega_3 = 2\Pi$ rad/sec, $\omega = 0.4\Pi$ rad/sec, $I_1 = 33/1.2$ m^{2}Kg, $l = 0.1730$ m, $g = 0.04$, the Amieux and Dureigne's optimal solution is $\zeta^* = 0.07897$ (from $\zeta^* = 2.8\ m|\omega-\omega_3|[\rho(\omega+\omega_3)/\omega]^{1/2}$) with 4 roots having the same real part -0.094035.

The true optimal solution is found to be of type I and the variation of the maximum real part with ζ is shown in Fig. 2. This proves that a wrong assumption on root distribution was used. The indirect optimization method proposed by Hughes[2,5] gives the optimal solution $\zeta^* = 0.07191$ and the maximum real part $= -0.070129$ as the solution to $T_2 = \partial T_2/\partial\zeta = 0$, where $T_2 = a_1{}^*a_2{}^*a_3{}^* - a_3{}^{*2} - a_1{}^{*2}a_4{}^*$.

References

[1] Amieux, J. C. and Dureigne, M., "Analytical Design of Optimal Nutation Dampers," *Journal of Spacecraft and Rockets*, Vol. 9, No. 12, Dec. 1972, pp. 934–935.

[2] Hughes, P. C. and Nguyen, P. K., "Minimum Time Decay in Linear Stationary Systems," *IEEE Transactions on Circuit Theory*, Vol. CT-19, No. 2, March 1972.

[3] Borrelli, R. L. and Leliakov, I. P., "An Optimization Technique for the Transient Response of Passively Stable Satellites," *Journal of Optimization Theory and Applications*, Vol. 10, No. 6, 1972, pp. 344–361.

[4] Nguyen, P. K., "A Practical Method for Optimum Transient Riddance in Damped Linear System," TN 151, 1970. Institute for Aerospace Studies, University of Toronto, Toronto, Ontario, Canada.

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Reply by Author to P. K. Nguyen

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P. K. NGUYEN brought to our attention the outstanding study done by R. L. Borrelli and I. P. Leliakov[1] published at the same time as our paper,[2] unknown to the authors. For the class of dynamical systems defined in Ref. 1, the problem is now completely solved. P. C. Hughes and P. K. Nguyen[3] utilize the variational approach to minimize the real part of the least damped root of the system. Let us compare the results of our approach and P. K. Nguyen's approach with Borrelli and

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Leliakov's one for the ball-in-tube damper case.[2] We recall[2] the system characteristic equation

$$P(\lambda) = \lambda^4 + \frac{q}{1-\rho}\lambda^3 + \left[\frac{r}{1-\rho} + \frac{\omega^2 - \rho\omega_3(\omega+\omega_3)}{1-\rho}\right]\lambda^2 + q\frac{\omega^2}{1-\rho}\lambda + r\frac{\omega^2}{1-\rho} - \rho\frac{\omega\omega_3{}^3}{1-\rho}$$

with

$$r = g\omega_3{}^2$$

Following the authors of Ref. 1, our problem fits their formulation with

$$\sigma_1 = q \qquad \sigma_2 = r$$

$$A_1 = \frac{1}{1-\rho}, \quad A_2 = \frac{\omega^2 - \rho\omega_3(\omega+\omega_3)}{1-\rho}, \quad A_3 = \frac{\omega^2}{1-\rho},$$

$$A_4 = -\frac{\rho\omega\omega_3{}^3}{1-\rho}$$

Note that Ref. 1, p. 357 should read

$$a_4 = Ha_2 - HA_2 + A_4$$

and therefore

$$G^2 \equiv HA_2 - A_4 - H^2$$

Now we verify the inequality $G < 2H$ so that, according to Ref. 1, the *only* optimum found is of type (2,2)—i.e., one double complex root. Now the least damped mode has its real part given exactly by

$$\delta = -G/2H^{1/2} = -\tfrac{1}{2}[\rho(\omega-\omega_3)^2(\omega+\omega_3)/\omega(1-\rho)]^{1/2}$$

and the optimal set of parameters is

$$q^* = 2(1-\rho)^{1/2}|\omega-\omega_3|\,[\rho(\omega+\omega_3)/\omega]^{1/2}$$
$$r^* = \omega^2 + \rho(\omega_3{}^3 - \omega^3/\omega)$$

Comparing these results with ours, recalling that with our parameters ρ is negligible compared to 1 ($\rho \simeq 2.25\times10^{-4}$), we find an amazing agreement.

The value of r shows that indeed the tuned conditions yield the optimum damping. We further point out that we were seeking a double root, and hence have expanded the Taylor's series to the'second-order term.

If we analyze now P. K. Nguyen's numerical results, we find: 1) a less optimal damping, $\delta = -0.070129$ as compared to our $\delta = -0.094035$; 2) a root configuration that does not agree with the optimal one as shown in Ref. 1 and that is not even one of the extraneous root geometries, which could be considered as local optima; and 3) no mention is made of the value of the second parameter available, namely r. The optimum must really be found with respect to two parameters, in most cases, as indicated by Borrelli and Leliakov.

This approach could also be used on the single-degree-of-freedom gyroscope. I take the opportunity here to correct the typographical errors[2] in the mathematical formulation, which should read as follows:

$$I(\dot\omega_1 + \omega\omega_2) - \xi\dot\alpha - \Gamma\alpha = 0$$
$$I\dot\omega_2 - (H + I\omega)\omega_1 - (H - J\omega_3)\dot\alpha = 0$$
$$J\ddot\alpha + \xi\dot\alpha + (\Gamma + H\omega_3)\alpha + J\dot\omega_1 + H\omega_2 = 0$$

The characteristic equation is now

$$P(\lambda) = \lambda^4 + \frac{\xi}{J}(1+\rho)\lambda^3 + \left[\omega(\omega+\rho h) - \rho(\omega_3 - h)(h-\omega) + h\omega_3 + \frac{\Gamma}{J}(1+\rho)\right]\lambda^2 + \frac{\xi}{J}(\omega+\rho h)^2\lambda + h\omega\omega_3(\omega+\rho h) + \frac{\Gamma}{J}(\omega+\rho h)^2$$

The problem is then well suited for the Borrelli and Leliakov method with

$$\sigma_1 = \xi, \quad \sigma_2 = \Gamma, \quad A_1 = \frac{1}{J}(1+\rho), \quad A_3 = \frac{1}{J}(\omega+\rho h)^2$$

$$A_2 = \omega(\omega+\rho h) + h\omega_3 - \rho(\omega_3 - h)(h-\omega), \quad A_4 = h\omega\omega_3(\omega+\rho h)$$

Now if we keep only the first-order term in ρ, we find

$$\delta = -\frac{1}{2}|\omega-h|\left[\rho\frac{\omega+\omega_3}{\omega+\rho(\omega+h)}\right]^{1/2}$$

and

$$q^* = 2|\omega-h|\left[\rho\frac{\omega+\omega_3}{\omega+\rho(\omega+h)}\right]^{1/2}$$

$$r^* = \frac{\Gamma}{J}(1+\rho) + h\omega_3 = \omega^2 + \rho A$$

where A is a positive expression in ω and h. These expressions are fully consistent with our results, and prove the validity of our assumptions and the efficiency of our approach, as far as small dampers for spinning spacecraft are concerned. Further work has been done using the same approach on two-degrees-of-freedom dampers, which lead to a sixth-order system with real characteristic coefficients that can be reduced to a third-order system with complex characteristic coefficients, for which Borrelli and Leliakov's method of analysis cannot be readily used.

References

[1] Borrelli, R. L. and Leliakov, I. P., "An Optimization Technique for the Transient Response of Passively Stable Satellites," *Journal of Optimization Theory and Applications*, Vol. 10, No. 6, Dec. 1972, pp. 344–351.

[2] Amieux, J. C. and Dureigne, M., "Analytical Design of Optimal Nutation Dampers," *Journal of Spacecraft and Rockets*, Vol. 9, No. 12, Dec. 1972, pp. 934–935.

[3] Hughes, P. C. and Nguyen, P. K., "Minimum Time Decay in Linear Stationary Systems," *IEEE Transactions on Circuit Theory*, Vol. CT-19, No. 2, March 1972.